

Degeneracy in finite time of 1D quasilinear wave equations II

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Abstract

We consider the large time behavior of solutions to the following nonlinear wave equation: $\partial_t^2 u = c(u)^2 \partial_x^2 u + \lambda c(u) c'(u) (\partial_x u)^2$ with the parameter $\lambda \in [0, 2]$. If $c(u(0, x))$ is bounded away from a positive constant, we can construct a local solution for smooth initial data. However, if $c(\cdot)$ has a zero point, then $c(u(t, x))$ can be going to zero in finite time. When $c(u(t, x))$ is going to 0 in finite time, the equation degenerates. We give a sufficient condition that the equation with $0 \leq \lambda < 2$ degenerates in finite time.

1 Introduction

In this paper, we consider the Cauchy problem of the following quasilinear wave equation:

$$\begin{cases} \partial_t^2 u = c(u)^2 \partial_x^2 u + \lambda c(u) c'(u) (\partial_x u)^2, & (t, x) \in (0, T] \times \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ \partial_t u(0, x) = u_1(x), & x \in \mathbb{R}, \end{cases} \quad (1.1)$$

where $u(t, x)$ is an unknown real valued function, $0 \leq \lambda \leq 2$ and $c'(\theta) = dc(\theta)/d\theta$. This parameterized equation has been introduced by Glassey, Hunter and Zheng [5]. As is explained later, this equation has different mathematical and physical backgrounds depending on λ (see also Chen and Shen [3]).

Throughout this paper, we assume that $c \in C^\infty((-1, \infty)) \cap C([-1, \infty))$ satisfies that

$$c(\theta) > 0 \quad \text{for all } \theta > -1, \quad (1.2)$$

$$c(-1) = 0, \quad (1.3)$$

$$c'(\theta) > 0 \quad \text{for all } \theta > -1. \quad (1.4)$$

Furthermore, we assume that there exists a constant $c_0 > 0$ such that

$$c(u_0(x)) \geq c_0 \quad (1.5)$$

for all $x \in \mathbb{R}$. A typical example of $c(\theta)$ satisfying (1.2)-(1.4) is $c(\theta) = (1 + \theta)^a$ with $a > 0$.

The assumptions (1.2) and (1.5) enable us to regard the equation in (1.1) as a strictly hyperbolic equation near $t = 0$. By the standard local existence theorem for strictly

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hyperbolic equations, the local solution of (1.1) with smooth initial data uniquely exists until the one of the following two phenomena occurs. The first one is the blow-up:

$$\overline{\lim}_{t \nearrow T^*} (\|\partial_t u(t)\|_{L^\infty} + \|\partial_x u(t)\|_{L^\infty}) = \infty.$$

The second is the degeneracy of the equation:

$$\lim_{t \nearrow T^*} \inf_{(s,x) \in [0,t] \times \mathbb{R}} c(u(s,x)) = 0.$$

When the equation degenerates, the standard local well-posedness theorem does not work since the equation loses the strict hyperbolicity. In general, for non-strictly hyperbolic equations, the persistence of the regularity of solutions does not hold (see Remark 1.5). The aim of this paper is to give a sufficient condition for the occurrence of the degeneracy of the equation with $0 \leq \lambda < 2$. The main theorem of this paper is the following.

Theorem 1.1. Let $0 \leq \lambda < 2$, $(u_0, u_1) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$ and $u_1 \not\equiv 0$. Suppose that the initial data (u_0, u_1) and c satisfy that (1.2)-(1.5) and

$$u_1(x) \pm c(u_0(x))\partial_x u_0(x) \leq 0 \quad \text{for all } x \in \mathbb{R}. \quad (1.6)$$

Then there exists $T^* > 0$ such that a solution of (1.1) exists uniquely and satisfies that $u \in C([0, T^*]; H^2(\mathbb{R})) \cap C^1([0, T^*]; H^1(\mathbb{R}))$ and

$$\lim_{t \nearrow T^*} \inf_{(s,x) \in [0,t] \times \mathbb{R}} c(u(s,x)) = 0. \quad (1.7)$$

Furthermore, if $0 < \lambda < 2$, then

$$\lim_{t \nearrow T^*} c(u(t, x_0)) = 0 \quad (1.8)$$

for some $x_0 \in \mathbb{R}$.

If $\lambda = 2$, then the equation in (1.1) is formally equivalent to the following conservation system:

$$\partial_t \begin{pmatrix} U \\ V \end{pmatrix} - \partial_x \begin{pmatrix} V \\ \int_{-1}^U c(\theta)^2 d\theta \end{pmatrix} = 0,$$

where $U(t, x) = u(t, x)$, $V(t, x) = \int_{-\infty}^x \partial_t u(t, y) dy$. This conservation system is called p-system and describes several phenomena of the wave propagation in nonlinear media including the electromagnetic wave in a transmission line, shearing-motion in elastic-plastic rods and 1 dimensional gas dynamics (see Ames and Lohner [1] and Zabusky [24]). In addition to the assumptions of Theorem 1.1, if $\int_{\mathbb{R}} u_1(x) dx > -2 \int_{-1}^0 c(\theta) d\theta$ is assumed, then (1.1) with $\lambda = 2$ has a global smooth solution such that the equation does not degenerate (e.g. Johnson [7] and Yamaguchi and Nishida [23]). On the other hand, in [18, 19] (see Remark 1.5 in [18] and Theorem 4.1 in [19]), the author has shown that the degeneracy (1.8) occurs in finite time, if $\int_{\mathbb{R}} u_1(x) dx < -2 \int_{-1}^0 c(\theta) d\theta$. Namely, these results say that $-2 \int_{-1}^0 c(\theta) d\theta$ is a threshold of $\int_{\mathbb{R}} u_1(x) dx$ separating the global existence of solutions (such that the equation does not degenerate) and the degeneracy of the equation

under the assumption (1.6). If (1.6) is not satisfied, then solutions can blow up in finite time (e.g. Klainerman and Majda [9], Manfrin [13] and Zabusky [24]). The main theorem of this paper implies that the degeneracy in finite time of the equation in (1.1) can occur regardless of $\int_{\mathbb{R}} u_1(x)dx$, when $0 \leq \lambda < 2$.

When $\lambda = 1$, the equation in (1.1) is called variational wave equation. As its name suggests, the equation with $\lambda = 1$ has a variational structure. The variational wave equation has some physical backgrounds including nematic liquid crystal and long waves on a dipole chain in the continuum limit (see [5]). In [4, 5], Glassey, Hunter and Zheng have shown that solutions can blow up in finite time, if (1.6) is not satisfied (see also Remark 1.4). There are a lot of papers devoted to the global existence of weak solutions to variational wave equations (e.g. Bressan and Zheng [2] and Zhang and Zheng [25, 26, 27]).

When $\lambda = 0$, the equation in (1.1) describes the wave of entropy in superfluids (e.g. Landau and Lifshitz [11]). This equation is the one dimensional version of

$$\partial_t^2 u = c(u)^2 \Delta u, \quad (t, x) \in (0, T] \times \mathbb{R}^3,$$

which has been studied in Lindblad [15]. In [15], Lindblad has shown that solutions exist globally in time with small initial data.

In [8, 17], Kato and the author have shown that the equation in (1.1) with $c(\theta) = 1 + \theta$ and $\lambda = 0, 1$ degenerates in finite time, if initial data are smooth, compactly supported and satisfy (1.5) and (1.6). The main theorem of this paper removes the compactness condition on initial data and extends the result in [8, 17] to (1.1) with more general $c(\theta)$ and $0 \leq \lambda < 2$. In [17, 18], the generalization on λ has already been pointed out without a proof. In fact, applying the method in [17, 18] to the equation in (1.1), we can generalize the result in [17, 18] to (1.1) with $0 \leq \lambda < 2$ and $c(\theta) = 1 + \theta$. However the compactness condition plays a crucial role in [8, 17], since we use the following estimates for bounded solutions under the assumption that initial data are compactly supported:

$$C(1+t) \geq \int_{\mathbb{R}} u(t, x) dx \tag{1.9}$$

and

$$C(1+t) \geq (2-\lambda) \int_0^t \int_0^s \int_{\mathbb{R}} c(u) c'(u) (\partial_x u(\tau, x))^2 dx d\tau ds. \tag{1.10}$$

(1.9) is formally shown by the finiteness of the propagation speed and the boundedness of solutions. (1.10) is shown by the equation in (1.1), (1.9) and the integration by parts. Furthermore, taking $c(u) = 1 + u$, we use the following estimate in [17, 18]:

$$-\int_{\mathbb{R}} u(t, x) dx \leq C(1+t) \left((1+t) \int_{\mathbb{R}} (1+u) (\partial_x u(\tau, x))^2 dx \right)^{\frac{1}{2}}, \tag{1.11}$$

which is shown by the fundamental theorem of calculus and the finiteness of the propagation speed. We can not obtain the above estimates directly, if initial data are not compactly supported. The first idea of the proof of Theorem 1.1 is the use of Riemann invariant, which is a major tool for the study of 2×2 conservation systems. When we use the Riemann invariant in the reduction from (1.1) to a first order system, $\int_{-\infty}^x c(u) c'(u) (\partial_x u(t, x))^2 dx$ appears as a force term in the first order system (see (3.1) and (3.2)). The second idea for

the proof is to divide the situation into the two cases that $\int_0^t \int_{\mathbb{R}} c(u)c'(u)(\partial_x u(s, x))^2 dx ds$ is bounded or not. If $\int_0^t \int_{\mathbb{R}} c(u)c'(u)(\partial_x u(s, x))^2 dx ds$ is bounded, then (1.10) holds and (1.9) can be shown by the use of the Riemann invariant. Hence we can use a variation of the method in [8, 17]. The key for the generalization on $c(\cdot)$ is the use of $\tilde{G}(u) = \int_{-1}^u \sqrt{c(\theta)c'(\theta)} d\theta$. We use $\tilde{G}(u)$ in order to generalize the estimate (1.11). If $\int_0^t \int_{\mathbb{R}} c(u)c'(u)(\partial_x u(s, x))^2 dx ds$ is not bounded, we can use the method in [18]. In the case that $\int_{\mathbb{R}} u_1(x) dx \notin L^1(\mathbb{R})$, the Riemann invariant can not be defined in general. Theorem 1.1 with $u_1 \notin L^1(\mathbb{R})$ can be shown by applying the same argument as in [18].

Remark 1.2. Addition to the assumptions of Theorem 1.1, if initial data are compactly supported, then (1.8) holds for (1.1) with $0 \leq \lambda < 2$, which can be shown by the finiteness of the propagation speed (see [8, 17]). Our method does not work for the case that $\lambda = 0$ (see Remark 3.2).

Remark 1.3. Under the assumptions (1.2)-(1.5), there is still no global existence result of (1.1) for $0 \leq \lambda < 2$. In stead of the assumptions, we assume that $c \in C^\infty(\mathbb{R})$ satisfies that

$$\begin{aligned} c_1 &\leq c(\theta) \leq c_2 \quad \text{for all } \theta \in \mathbb{R}, \\ c'(\theta) &\geq 0 \quad \text{for all } \theta \in \mathbb{R} \end{aligned}$$

for some positive constants c_1 and c_2 . Under these assumptions and (1.6), Zhang and Zheng [25] have shown that (1.1) has global smooth solutions with $\lambda = 1$. This global existence result has been extended to $0 \leq \lambda \leq 2$ in the author's paper [17].

Remark 1.4. Here we collect some open questions for (1.1). When $\lambda = 1$ and 2, it is known that blow-up solutions exist, if (1.6) is not satisfied. From the first and the second equations in (2.4), we can expect that blow-up solutions exist for $0 < \lambda \leq 2$, if (1.6) is not satisfied, since the right hand sides of the first and the second equations in (2.4) contain λR^2 and λS^2 respectively, which seems to derive the singularity formation. However, the existence of the blow-up solution is still open, since the proofs of the blow-up theorems for $\lambda = 1$ and 2 rely on structures of the equation. When $\lambda = 0$, if $c(\cdot)$ is uniformly positive, then it seems possible that (1.1) has global smooth solution for any smooth initial data, although a complete proof or a counterexample for this problem is also open.

Remark 1.5. It is known that a loss of the regularity appears for solutions to the following non-strictly hyperbolic equation:

$$\partial_t^2 u - t^{2l} \partial_x^2 u - h t^{l-1} \partial_x u = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R},$$

where h is a constant and $l \in \mathbb{N}$. Namely, in general, $(u, \partial_t u)$ does not belong to $C^1([0, \infty), H^s(\mathbb{R})) \times C([0, \infty), H^{s-1}(\mathbb{R}))$ with $(u(0, x), \partial_t u(0, x)) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ (see Taniguchi and Tozaki [20], Yagdjian [22] and Qi [16]). From this fact, we can expect that solutions of (1.1) have a singularity when the equation degenerates.

This paper is organized as follows: In Section 2, we recall the local well-posedness and some properties of solutions of (1.1). In Sections 3 and 4, we show Theorem 1.1 in the cases that $u_1 \in L^1(\mathbb{R})$ and $u_1 \notin L^1(\mathbb{R})$ respectively.

Notation

We denote Lebesgue space for $1 \leq p \leq \infty$ and L^2 Sobolev space with the order $m \in \mathbb{N}$ on \mathbb{R} by $L^p(\mathbb{R})$ and $H^m(\mathbb{R})$. For a Banach space X , $C^j([0, T]; X)$ denotes the set of functions $f : [0, T] \rightarrow X$ such that $f(t)$ and its k times derivatives for $k = 1, 2, \dots, j$ are continuous. Various positive constants are simply denoted by C .

2 Preliminary

We recall the local well-posedness of (1.1) and some properties of solutions of (1.1). By applying the well-known local well-posedness Theorem (e.g Hughes, Kato and Marsden [6], Majda [12] or Taylor [21]), we can obtain the following theorem.

Theorem 2.1. Let $\lambda \in \mathbb{R}$. Suppose that $(u_0, u_1) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$ and that (1.2) and (1.5) hold. Then there exist $T > 0$ and a unique solution u of (1.1) with

$$u \in \bigcap_{j=0,1,2} C^j([0, T]; H^{2-j}(\mathbb{R})) \quad (2.1)$$

and

$$c(u(t, x)) \geq \delta(T) \quad \text{for } (t, x) \in [0, T] \times \mathbb{R}, \quad (2.2)$$

where $\delta(T)$ is a positive monotone decreasing function of $T \in [0, \infty)$. Furthermore, if (1.1) does not have a global solution u satisfying (2.1) and (2.2), then the solution u satisfies

$$\overline{\lim}_{t \nearrow T^*} (\|\partial_t u(t)\|_{L^\infty} + \|\partial_x u(t)\|_{L^\infty}) = \infty$$

or

$$\lim_{t \nearrow T^*} \inf_{(s, x) \in [0, t] \times \mathbb{R}} c(u(s, x)) = 0$$

for some $T^* > 0$.

We denote the maximal existence time of the solution u of (1.1) constructed in Theorem 2.1 by T^* , that is,

$$T^* = \sup \{ T > 0 \mid \sup_{[0, T]} \{ \|\partial_t u(t)\|_{L^\infty} + \|\partial_x u(t)\|_{L^\infty} \} < \infty, \\ \inf_{[0, T] \times \mathbb{R}} c(u(t, x)) > 0 \}.$$

We set $R(t, x)$ and $S(t, x)$ as follows

$$\begin{cases} R = \partial_t u + c(u) \partial_x u, \\ S = \partial_t u - c(u) \partial_x u. \end{cases} \quad (2.3)$$

The functions R and S have been used in Glassey, Hunter and Zheng [4, 5] and Zhang and Zheng [25]. We recall some properties of R and S proved in [17].

By (1.1), R and S are solutions to the system of the following first order equations:

$$\begin{cases} \partial_t R - c(u) \partial_x R = \frac{c'(u)}{2c(u)} (RS - S^2) + \lambda \frac{c'(u)}{4c(u)} (R - S)^2, \\ \partial_x u = \frac{1}{2c(u)} (R - S), \\ \partial_t S + c(u) \partial_x S = \frac{c'(u)}{2c(u)} (SR - R^2) + \lambda \frac{c'(u)}{4c(u)} (S - R)^2. \end{cases} \quad (2.4)$$

Lemma 2.2. Let $0 \leq \lambda \leq 2$. Suppose that the assumptions of Theorem (1.1) are satisfied. Then we have

$$R(t, x), S(t, x) \leq 0 \text{ for } (t, x) \in [0, T^*) \times \mathbb{R}, \quad (2.5)$$

where R and S are the functions in (2.3) for the solution u of (1.1) constructed by Theorem 2.1.

Lemma 2.3. Let $p \geq \max\{2, \frac{2}{\lambda}\}$. Suppose that the assumptions of Theorem (1.1) are satisfied. Then we have for $0 < \lambda \leq 2$

$$\|R(t)\|_{L^p}^p + \|S(t)\|_{L^p}^p \leq \|R(0)\|_{L^p}^p + \|S(0)\|_{L^p}^p, \text{ for } t \in [0, T^*), \quad (2.6)$$

where R and S are the functions in (2.3) for the solution u of (1.1) constructed by Theorem 2.1. Furthermore $\|R(t)\|_{L^\infty}$ and $\|S(t)\|_{L^\infty}$ are uniformly bounded with $t \in [0, T^*)$ for $0 \leq \lambda \leq 2$.

Lemmas 2.2 and 2.3 have been shown in the author's paper [17]. The proofs are essentially the same as in the case that $\lambda = 1$, which are proved in Zhang and Zheng [25]. In [17, 18], it is assumed only $p \geq 2/\lambda$ for the inequality (2.6). But the proof in [17] is not collect for $p < 2$. In fact, in [17], the proof of (2.6) is based on the following inequality:

$$\frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}} \tilde{R}^p + \tilde{S}^p dx \leq -\left(\frac{1}{2} - \frac{\lambda}{4}\right) \int_{\mathbb{R}} \frac{c'(u)}{c(u)} \tilde{R} \tilde{S} (\tilde{R} - \tilde{S}) ((\tilde{R})^{p-2} - (\tilde{S})^{p-2}) dx,$$

where $\tilde{R} = -R$ and $\tilde{S} = -S$. If $p < 2$, the right hand side of this inequality is not negative except for $\lambda = 2$. However, in [17], we only use (2.6) for $p \geq 2$. If $\lambda = 2$, (2.6) holds for all $p \geq 1$.

We note that (2.5) implies that $\partial_t u(t, x) \leq 0$ for all $(t, x) \in [0, T^*) \times \mathbb{R}$.

3 Proof of Theorem 1.1 with $u_1 \in L^1(\mathbb{R})$

We show Theorem 1.1 in the case that $u_1 \in L^1(\mathbb{R})$.

First, we show (1.7) by the contradiction argument. From Theorem 2.1 and Lemma 2.3, it is enough to show that $T^* < \infty$. We set $G(u) = \int_{-1}^u c(\theta) d\theta$ for $u \geq -1$ and $\mu = 2 - \lambda$ and define the Riemann invariants $(w_1(t, x), w_2(t, x))$ and $(v_1(t, x), v_2(t, x))$ as follows:

$$\begin{aligned} w_1 &= \int_{-\infty}^x \partial_t u dx + G(u), \\ w_2 &= \int_{-\infty}^x \partial_t u dx - G(u) \end{aligned}$$

and

$$\begin{aligned} v_1 &= \int_x^\infty \partial_t u dx - G(u), \\ v_2 &= \int_x^\infty \partial_t u dx + G(u). \end{aligned}$$

From (1.1), $(w_1(t, x), w_2(t, x))$ and $(v_1(t, x), v_2(t, x))$ satisfy that the following systems:

$$\begin{cases} \partial_t w_1 - c(u) \partial_x w_1 = -\mu \int_{-\infty}^x \tilde{e}(t, y) dy, \\ \partial_t w_2 + c(u) \partial_x w_2 = -\mu \int_{-\infty}^x \tilde{e}(t, y) dy \end{cases} \quad (3.1)$$

and

$$\begin{cases} \partial_t v_1 - c(u) \partial_x v_1 = -\mu \int_x^\infty \tilde{e}(t, y) dy, \\ \partial_t v_2 + c(u) \partial_x v_2 = -\mu \int_x^\infty \tilde{e}(t, y) dy, \end{cases} \quad (3.2)$$

where $\tilde{e}(t, y) = c'(u)c(u)(\partial_x u)^2(t, y)$. Let $x_\pm(t)$ be characteristic curves on the first and third equations of (3.1) respectively. That is, $x_\pm(t)$ are solutions to the following differential equations:

$$\frac{d}{dt} x_\pm(t) = \pm c(u(t, x_\pm(t))). \quad (3.3)$$

(3.1) and (3.2) imply that

$$w_1(t, x_-(t)) = w_1(t_0, x_-(t_0)) - \mu \int_{t_0}^t \int_{-\infty}^{x_-(s)} \tilde{e}(s, y) dy ds \quad (3.4)$$

and

$$w_2(t, x_+(t)) = w_2(t_0, x_+(t_0)) - \mu \int_{t_0}^t \int_{-\infty}^{x_+(s)} \tilde{e}(s, y) dy ds. \quad (3.5)$$

Case that $\int_0^t \int_{\mathbb{R}} \tilde{e}(s, y) dy ds$ is bounded.

By the contradiction argument, we show that T^* is finite in the case that $\int_0^t \int_{\mathbb{R}} \tilde{e}(s, y) dy ds$ is bounded on $[0, \infty)$. We suppose that $T^* = \infty$. (3.4) and (3.5) imply that

$$w_1(t, x_-(t)) \geq w_1(0, x_-(0)) - \mu \int_0^\infty \int_{-\infty}^{x_-(s)} \tilde{e}(s, y) dy ds \quad (3.6)$$

and

$$w_2(t, x_+(t)) \geq w_2(0, x_+(0)) - \mu \int_0^\infty \int_{-\infty}^{x_+(s)} \tilde{e}(s, y) dy ds. \quad (3.7)$$

We fix an arbitrary number $\varepsilon > 0$. Since $\lim_{|x| \rightarrow \infty} u_0(x) = 0$, $u_1 \in L^1(\mathbb{R})$ and $\partial_x w_j(0, x) \leq 0$ for $j = 1, 2$, there exists a constant $M_0 > 0$ such that

$$G(0) - \varepsilon \leq w_1(0, x) \leq G(0)$$

and

$$-G(0) - \varepsilon \leq w_2(0, x) \leq -G(0)$$

for any $x \leq -M_0$. Noting $x_\pm(t)$ goes to $-\infty$ as $x_\pm(0) \rightarrow -\infty$ for all $t \geq 0$, since $\int_0^\infty \int_{\mathbb{R}} \tilde{e}(s, y) dy ds$ is bounded, we have by the Lebesgue convergence theorem

$$\lim_{x_\pm(0) \rightarrow -\infty} \int_0^\infty \int_{-\infty}^{x_\pm(s)} \tilde{e}(s, y) dy ds = 0.$$

Hence, from (3.6) and (3.7), there exists a constant $M_1 > 0$ such that if $x_{\pm}(0) \leq -\max\{M_0, M_1\}$, then for all $t \geq 0$

$$G(0) - \varepsilon \leq w_1(t, x_-(t)) \leq G(0)$$

and

$$-G(0) - \varepsilon \leq w_2(t, x_+(t)) \leq -G(0).$$

We note the positive constant M_1 can be chosen independently of t . Hence the equality $2G(u(t, x)) = w_1(t, x) - w_2(t, x)$ yields that

$$G(0) - \varepsilon \leq G(u(t, x)) \leq G(0) + \varepsilon,$$

if $x \leq x_-(t)$, where $x_-(0) \leq -\max\{M_0, M_1\}$. Since G is invertible and G^{-1} is continuous, this inequality implies that

$$|u(t, x)| \leq C\varepsilon \quad (3.8)$$

with $x \leq x_-(t)$, where $x_-(0) \leq -\max\{M_0, M_1\}$. From the above estimates of w_1 and $G(u)$, we have

$$-C\varepsilon \leq \int_{-\infty}^{x_-(t)} \partial_t u(t, y) dy \leq 0.$$

Since

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^{x_-(t)} u(t, x) - u_0(x) dx &= \int_{-\infty}^{x_-(t)} \partial_t u(t, x) dx \\ &\quad - (u(t, x_-(t)) - u_0(x_-(t)))c(u(t, x_-(t))), \end{aligned}$$

if $x_-(0) \leq -\max\{M_0, M_1\}$, then we have from (3.8)

$$- \int_{-\infty}^{x_-(t)} u(t, y) - u_0(x) dy \leq C\varepsilon t. \quad (3.9)$$

By using (3.2), we have in the same way as in the derivation of (3.9)

$$- \int_{x_+(t)}^{\infty} u(t, y) - u_0(x) dy \leq C\varepsilon t, \quad (3.10)$$

if $x_+(0) \geq M_2$ for sufficiently large $M_2 > 0$.

We set $F(t) = -\int_{\mathbb{R}} u(t, x) - u_0(x) dx$ and take $M \geq \max\{M_0, M_1, M_2\}$. From the integration by parts and (1.1), it follows that $F''(t) = \mu \int_{\mathbb{R}} \tilde{e}(t, x) dx \geq 0$. Integrating this equality twice on $[0, t]$ and dividing by t , we have

$$\frac{F(t)}{t} \geq F'(0).$$

By (3.23) and (3.10), we have

$$\begin{aligned} F'(0) &\leq \frac{F(t)}{t} = \frac{-1}{t} \left(\int_{-\infty}^{x_-(t)} + \int_{x_-(t)}^{x_+(t)} + \int_{x_+(t)}^{\infty} \right) u(t, x) - u_0(x) dx \\ &\leq C\varepsilon - \frac{1}{t} \int_{x_-(t)}^{x_+(t)} u(t, x) - u_0(x) dx. \end{aligned} \quad (3.11)$$

Now we estimate the second term of the right hand side of (3.11). We set $\tilde{G}(u) = \int_{-1}^u \sqrt{c(\theta)c'(\theta)}d\theta$. The Schwarz inequality implies that

$$\tilde{G}(u)^2 \leq \int_{-1}^u c'(\theta)d\theta \int_{-1}^u c(\theta)d\theta = c(u) \int_{-1}^u c(\theta)d\theta.$$

Hence $\tilde{G}(u)$ can be defined for $u \geq -1$. From (1.2) and (1.3), we have that $\tilde{G}'(u) = \sqrt{c(u)c'(u)} > 0$, from which $\tilde{G}(\cdot)$ is invertible on $[0, \infty)$ and $\tilde{G}^{-1}(\cdot)$ is continuous. The fundamental theorem of calculus yields that

$$\tilde{G}(u(t, x)) = \tilde{G}(u(t, x_-(t))) + \int_{x_-(t)}^x \sqrt{c(u)c'(u)} \partial_y u(t, y) dy. \quad (3.12)$$

Applying (3.8) to the first term of the right hand side of (3.12) and the Schwarz inequality to the second term, we have

$$\tilde{G}(u(t, x)) \geq \tilde{G}(0) - C\varepsilon - \sqrt{|x_+(t) - x_-(t)| \int_{\mathbb{R}} \tilde{e}(t, y) dy}$$

for $x \in [x_-(t), x_+(t)]$. Since \tilde{G}^{-1} is a monotone increasing function, we have

$$u(t, x) \geq \tilde{G}^{-1} \left(\tilde{G}(0) - C\varepsilon - \sqrt{|x_+(t) - x_-(t)| \int_{\mathbb{R}} \tilde{e}(t, y) dy} \right).$$

From (3.3) and (3.8), we have $|x_+(t) - x_-(t)| \leq C_M + C^*t$, where $C_M > 0$ depends on M_j for $j = 1, 2, 3$ and $C^* > 0$ can be chosen independently of the three constants. Integrating the both sides of this inequality on $[x_-(t), x_+(t)]$, we have

$$\begin{aligned} - \int_{x_-(t)}^{x_+(t)} u(t, x) dx &\leq - (C_M + C^*t) \tilde{G}^{-1} \left(\tilde{G}(0) - C\varepsilon \right. \\ &\quad \left. - \sqrt{(C_M + C^*t) \int_{\mathbb{R}} \tilde{e}(t, y) dy} \right). \end{aligned} \quad (3.13)$$

While the Schwarz inequality implies that

$$\int_{x_-(t)}^{x_+(t)} |u_0(x)| dx \leq C \sqrt{C_M + C^*t} \|u_0\|_{L^2}.$$

From this inequality, (3.11) and (3.13), we have

$$\begin{aligned} F'(0) &\leq \frac{C \sqrt{C_M + C^*t} \|u_0\|_{L^2}}{t} + C\varepsilon \\ &\quad - \frac{C_M + C^*t}{t} \tilde{G}^{-1} \left(\tilde{G}(0) - C\varepsilon - \sqrt{(C_M + C^*t) \int_{\mathbb{R}} \tilde{e}(t, y) dy} \right). \end{aligned} \quad (3.14)$$

Since we assume that $\int_0^\infty \int_{\mathbb{R}} \tilde{e}(s, y) dy ds < \infty$, there exists a monotone increasing sequence $\{t_j\}_{j \in \mathbb{N}}$ such that $\lim_{j \rightarrow \infty} t_j = \infty$ and

$$\lim_{j \rightarrow \infty} (C_M + C^*t_j) \int_{\mathbb{R}} \tilde{e}(t_j, y) dy = 0.$$

Putting $t = t_j$ in (3.14) and taking $j \rightarrow \infty$, since \tilde{G} is continuous, we obtain

$$F'(0) \leq C\varepsilon - C^* \tilde{G}^{-1} \left(\tilde{G}(0) - C\varepsilon \right) \leq C\varepsilon,$$

which contradicts to the assumption that $u_1 \not\equiv 0$, if ε is sufficiently small. Therefore we obtain $T^* < \infty$ in the case that $\int_0^t \int_{\mathbb{R}} \tilde{e}(s, y) dy ds$ is bounded.

Remark 3.1. The strictly positivity of c' is only used in the estimate of \tilde{G} . It is enough to assume that $c'(\theta) \geq 0$ for $\theta > 0$ in the case that $\int_0^t \int_{\mathbb{R}} \tilde{e}(s, y) dy ds$ is unbounded. In this case, we use the method in [18, 19]. In [18, 19], instead of (1.4), it is assumed that $c'(\theta) \geq 0$ for the occurrence of the degeneracy of the equation in (1.1) with $\lambda = 2$.

Case that $\int_0^t \int_{\mathbb{R}} \tilde{e}(s, y) dy ds$ is unbounded.

We suppose that $T^* = \infty$. In this case, from the identity

$$-\int_{\mathbb{R}} \partial_t u(t, x) dx = -\int_{\mathbb{R}} u_1(x) dx + \mu \int_0^t \int_{\mathbb{R}} \tilde{e}(s, x) dx ds,$$

there exists a positive number T such that

$$-\int_{\mathbb{R}} \partial_t u(T, x) dx > 2G(0). \quad (3.15)$$

From (3.6) and (3.7), if the plus and minus characteristic curves cross at some point (t_0, x_0) with $t_0 \geq T$, then we have by the definitions of w_1 and w_2

$$\begin{aligned} 2G(u(t_0, x_0)) &= w_1(T, x_-(T)) - w_2(T, x_+(T)) - \mu \int_T^{t_0} \int_{x_+(s)}^{x_-(s)} \tilde{e}(s, y) dy ds \\ &= \int_{x_+(T)}^{x_-(T)} \partial_t u(T, x) dx + G(u(T, x_+(T))) + G(u(T, x_-(T))) \\ &\quad - \mu \int_T^{t_0} \int_{x_+(s)}^{x_-(s)} \tilde{e}(s, y) dy ds. \end{aligned} \quad (3.16)$$

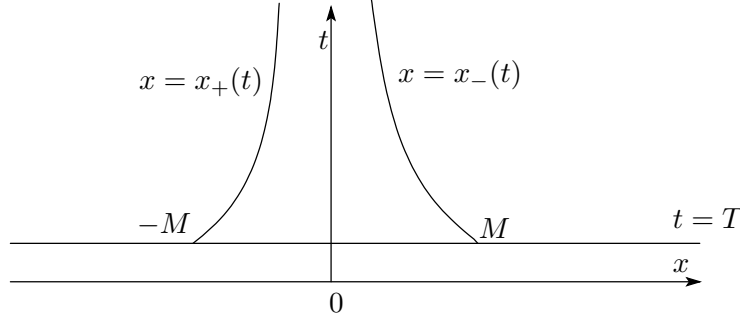
By (3.15) and the facts that $\lim_{|x| \rightarrow \infty} u(T, x) = 0$ and that $\partial_t u(T, \cdot) \in L^1(\mathbb{R})$, there exists a number $M > 0$ such that

$$\int_{-M}^M \partial_t u(T, x) dx + G(u(T, -M)) + G(u(T, M)) < 0. \quad (3.17)$$

We set $\tilde{F}(t) = -\int_{\mathbb{R}} u(t, x) - u(T, x) dx$. We derive a estimate of $\tilde{F}(t)$ which contradicts to (3.17).

Suppose that the plus and minus characteristic curves $x_{\pm}(t)$ defined in (3.3) pass through $(T, \mp M)$ respectively. The characteristics $x_{\pm}(t)$ are drawn on the (x, t) plane as follows:

Figure 1: the two characteristic curves on the (x, t) plane



From (3.16) and (3.17), these characteristic curves $x_+(t)$ and $x_-(t)$ do not cross for all $t \geq T$. Hence it follows that

$$\lim_{t \rightarrow \infty} c(u(t, x_{\pm}(t))) = 0. \quad (3.18)$$

And \tilde{F} can be divided as follows:

$$\tilde{F}(t) = - \left(\int_{-\infty}^{x_+(t)} + \int_{x_+(t)}^{x_-(t)} + \int_{x_-(t)}^{\infty} \right) u(t, x) - u(T, x) dx. \quad (3.19)$$

Now we estimate $\int_{-\infty}^{x_+(t)} u(t, x) - u(T, x) dx$. From (3.3), we have

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^{x_+(t)} u(t, x) - u(T, x) dx &= \int_{-\infty}^{x_+(t)} \partial_t u(t, x) dx \\ &\quad + (u(t, x_+(t)) - u(T, x_+(t)))c(u(t, x_+(t))). \end{aligned} \quad (3.20)$$

From (3.5) and the definition of w_2 , the first term of the right hand side of (3.20) can be estimated as follows:

$$\begin{aligned} \int_{-\infty}^{x_+(t)} \partial_t u(t, x) dx &= G(u(t, x_+(t))) + w_2(T, -M) - \mu \int_T^t \int_{-\infty}^{x_+(s)} \tilde{e}(s, x) dx ds \\ &\geq w_2(T, -M) - \mu \int_T^t \int_{-\infty}^{x_+(s)} \tilde{e}(s, x) dx ds. \end{aligned}$$

From the boundedness of u , the second term of the right hand side of (3.20) can be estimated as

$$(u(t, x_+(t)) - u(T, x_+(t)))c(u(t, x_+(t))) \geq -Cc(u(t, x_+(t))).$$

Hence we have from (3.20)

$$\begin{aligned} \int_{-\infty}^{x_+(t)} u(t, x) - u(T, x) dx &\geq (t - T)w_2(T, -M) - C \int_T^t c(u(s, x_+(s))) ds \\ &\quad - \mu \int_T^t \int_T^s \int_{-\infty}^{x_+(\tau)} \tilde{e}(\tau, x) dx d\tau ds. \end{aligned} \quad (3.21)$$

Using v_1 instead of w_2 , in the same way as in the derivation of (3.21), we get

$$\begin{aligned} \int_{x_-(t)}^{\infty} u(t, x) - u(T, x) dx &\geq (t - T)v_1(T, -M) - C \int_T^t c(u(s, x_-(s))) ds \\ &\quad - \mu \int_T^t \int_T^s \int_{x_-(\tau)}^{\infty} \tilde{e}(\tau, x) dx d\tau ds. \end{aligned} \quad (3.22)$$

The boundedness of $u(t, x)$ and $|x_+(t) - x_-(t)|$ yields that

$$- \int_{x_+(t)}^{x_-(t)} u(t, x) - u(T, x) dx \leq C|x_-(t) - x_+(t)| \leq C. \quad (3.23)$$

While, in the same way as in the computation of F , we have

$$\tilde{F}(t) = (t - T)\tilde{F}'(T) + \mu \int_T^t \int_T^s \int_{\mathbb{R}} \tilde{e}(\tau, x) dx d\tau ds. \quad (3.24)$$

By (3.19), (3.21), (3.22) and (3.23), we have

$$\begin{aligned} \tilde{F}(t) &\leq C - (t - T)(w_2(T, -M) + v_1(T, M)) \\ &\quad + C \int_T^t c(u(s, x_+(s))) + c(u(s, x_-(s))) ds \\ &\quad + \mu \int_T^t \int_T^s \left(\int_{-\infty}^{x_+(\tau)} + \int_{x_+(\tau)}^{\infty} \right) \tilde{e}(\tau, x) dx d\tau ds. \end{aligned} \quad (3.25)$$

From the definitions of w_2 and v_1 , the second term of the right hand side of (3.25) can be written as follows:

$$\begin{aligned} w_2(T, -M) + v_1(T, M) &= \left(\int_{-\infty}^{-M} + \int_M^{\infty} \partial_t u(T, x) dx \right) \\ &\quad - (G(u(T, M)) + G(u(T, -M))), \end{aligned}$$

from which, (3.24) and (3.25) yield that

$$\begin{aligned} - \int_{-M}^M \partial_t u(T, x) dx &\leq \frac{-\mu}{(t - T)} \int_T^t \int_T^s \int_{x_+(\tau)}^{x_-(\tau)} \tilde{e}(\tau, x) dx d\tau ds \\ &\quad G(u(T, M)) + G(u(T, -M)) \\ &\quad + \frac{C}{t - T} \int_T^t c(u(s, x_+(s))) + c(u(s, x_-(s))) ds \\ &\leq G(u(T, M)) + G(u(T, -M)) \\ &\quad + \frac{C}{t - T} \int_T^t c(u(s, x_+(s))) + c(u(s, x_-(s))) ds. \end{aligned}$$

From (3.18), the second term of the right hand side of the above inequality tends to 0 as $t \rightarrow \infty$. Hence, taking $t \rightarrow \infty$ in the above inequality, we have

$$- \int_{-M}^M \partial_t u(T, x) dx \leq G(u(T, M)) + G(u(T, -M)),$$

which contradicts to (3.17). Hence we have that $T^* < \infty$ in the case $\int_0^t \int_{\mathbb{R}} \tilde{e}(s, y) dy ds$ is unbounded.

From the above argument of the two cases, we have $T^* < \infty$ and (1.7).

Next we give an outline of the proof of (1.8) for $0 < \lambda < 2$. The proof is the same as in [8, 17, 18]. Since $u(t, x)$ is a monotone decreasing function with t for all $x \in \mathbb{R}$, we can define $\tilde{u}(x) = \lim_{t \nearrow T^*} u(t, x)$. While, (2.6) in Lemma 2.3 implies that $\|c(u)\partial_x u(t)\|_{L^p}$ is uniformly bounded with $t \in [0, T^*)$ and $p = \max\{2, 2/\lambda\}$. Hence, by the standard argument on the Sobolev space, it follows that $G(\tilde{u}) - G(0), c(\tilde{u})\partial_x \tilde{u} \in L^p(\mathbb{R})$ and that $G(\tilde{u}(\cdot))$ is a continuous function. Therefore we have $\lim_{|x| \nearrow \infty} G(\tilde{u}) - G(0) = 0$, from which, the continuity of $G(\tilde{u}(\cdot))$ implies that $G(\tilde{u}(x_0)) = 0$ for some $x_0 \in \mathbb{R}$. While, from the monotonicity of $G(u(t, x))$ with t , we have

$$\lim_{t \nearrow T^*} \inf_{(s, x) \in [0, t] \times \mathbb{R}} G(u(s, x)) = \inf_{x \in \mathbb{R}} \lim_{t \nearrow T^*} G(u(t, x)) = \inf_{x \in \mathbb{R}} G(\tilde{u}(x)).$$

Hence, by the continuity of G^{-1} , we have $\lim_{t \nearrow T^*} u(t, x_0) = -1$, which implies (1.8).

Remark 3.2. In the case that $\lambda = 0$, since the boundedness of $\|c(\tilde{u})\partial_x \tilde{u}(t)\|_{L^p}$ is unknown for $p \neq \infty$, the above argument does not work. Hence the case that $\lambda = 0$ is excluded in (1.8).

4 Proof of Theorem 1.1 with $u_1 \notin L^1(\mathbb{R})$

By using the same argument as in [18], we can show Theorem 1.1 with $u_1 \notin L^1(\mathbb{R})$. We show that the degeneracy (1.7) occurs in finite time for the reader's convenience. (1.8) can be shown by same way as in the case that $u_1 \in L^1(\mathbb{R})$. In the same argument as in Section 3, we can say that then (1.7) occurs at T^* , if $T^* < \infty$. Hence it is enough to show that T^* is finite. We define a cut-off function $\psi \in C_0^\infty(\mathbb{R})$ as

$$\psi(x) = \begin{cases} 1 & \text{for } |x| \leq 1, \\ 0 & \text{for } |x| \geq 2 \end{cases}$$

and $0 \leq \psi(x) \leq 1$. We set $\psi_\varepsilon(x) = \psi(\varepsilon x)$ and $F_\varepsilon(t) = -\int_{\mathbb{R}} \psi_\varepsilon(x) u(t, x) dx$. From (1.1) and the integration by parts, we have

$$\begin{aligned} F_\varepsilon''(t) &= -\varepsilon^2 \int_{\mathbb{R}} \psi''(\varepsilon x) G_2(u) dx + \mu \int_{\mathbb{R}} \psi_\varepsilon(x) c(u) c'(u) (\partial_x u)^2 dx \\ &\geq -\varepsilon^2 \int_{\mathbb{R}} \psi''(\varepsilon x) G_2(u) dx, \end{aligned}$$

where $G_2(u) = \int_{-1}^u c(\theta)^2 d\theta$.

Since $-1 \leq u(t, x) \leq u(0, x) \leq C$, it follows that

$$F_\varepsilon''(t) \geq -C\varepsilon.$$

Namely we have

$$F_\varepsilon(t) \geq F_\varepsilon(0) + tF_\varepsilon'(0) - C\varepsilon t^2.$$

The boundedness of $u(t, x)$ with $(t, x) \in [0, \infty) \times \mathbb{R}$ yields that

$$|F_\varepsilon(t)| \leq \int_{\mathbb{R}} |\psi_\varepsilon(x)| |u(t, x)| dx \leq \frac{C}{\varepsilon}.$$

Hence we have

$$\frac{C}{\varepsilon} + C\varepsilon t^2 \geq tF'_\varepsilon(0).$$

Putting $\varepsilon = 1/(1+t)$, we have

$$\frac{2C(t+1)}{t} \geq - \int_{\mathbb{R}} \psi\left(\frac{x}{t+1}\right) u_1(x) dx.$$

Since $u_1 \notin L^1(\mathbb{R})$, the right hand side is going to infinity as $t \rightarrow \infty$, which is a contradiction. Therefore we have $T^* < \infty$.

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